Chapter 1
Wigner Distribution in Optics

Martin J. Bastiaans
Technische Universiteit Eindhoven, Faculteit Elektrotechniek,
Postbus 513, 5600 MB Eindhoven, Netherlands

1.1 Introduction
1.2 Elementary description of optical signals and systems
   1.2.1 Impulse response and coherent point-spread function
   1.2.2 Mutual coherence function and cross-spectral density
   1.2.3 Some basic examples of optical signals
1.3 Wigner distribution and ambiguity function
   1.3.1 Definitions
   1.3.2 Some basic examples again
   1.3.3 Gaussian light
   1.3.4 Local frequency spectrum
1.4 Some properties of the Wigner distribution
   1.4.1 Inversion formula
   1.4.2 Shift covariance
   1.4.3 Radiometric quantities
   1.4.4 Instantaneous frequency
   1.4.5 Moyal’s relationship
1.5 One-dimensional case and the fractional Fourier transformation
   1.5.1 Fractional Fourier transformation
   1.5.2 Rotation in phase space
   1.5.3 Generalized marginals – Radon transform
1.6 Propagation of the Wigner distribution
   1.6.1 First-order optical systems – ray transformation matrix
   1.6.2 Phase-space rotators – more rotations in phase space
   1.6.3 More general systems – ray-spread function
   1.6.4 Geometric-optical systems
   1.6.5 Transport equations
1.7 Wigner distribution moments in first-order optical systems
   1.7.1 Moment invariants
Chapter 1. Wigner Distribution in Optics

1.7.2 Moment invariants for phase-space rotators
1.7.3 Symplectic moment matrix – the bilinear ABCD law
1.7.4 Measurement of the moments
1.8 Coherent signals and the Cohen class
   1.8.1 Multi-component signals – auto-terms and cross-terms
   1.8.2 One-dimensional case and some basic Cohen kernels
   1.8.3 Rotation of the kernel
   1.8.4 Rotated version of the smoothed interferogram
1.9 Conclusion

References

1.1 Introduction

In 1932 Wigner\(^1\) introduced a distribution function in mechanics that permitted a description of mechanical phenomena in a phase space. Such a Wigner distribution was introduced in optics by Dolin\(^2\) and Walther\(^3,4\) in the sixties, to relate partial coherence to radiometry. A few years later, the Wigner distribution was introduced in optics again\(^5-11\) (especially in the area of Fourier optics), and since then, a great number of applications of the Wigner distribution have been reported.

While the mechanical phase space is connected to classical mechanics, where the movement of particles is studied, the phase space in optics is connected to geometrical optics, where the propagation of optical rays is considered. And where the position and momentum of a particle are the two important quantities in mechanics, in optics we are interested in the position and the direction of an optical ray. We will see that the Wigner distribution represents an optical field in terms of a ray picture, and that this representation is independent of whether the light is partially coherent or completely coherent.

We will observe that a description by means of a Wigner distribution is in particular useful when the optical signals and systems can be described by quadratic-phase functions, i.e., when we are in the realm of first-order optics: spherical waves, thin lenses, sections of free space in the paraxial approximation, etc. Although formulated in Fourier-optical terms, the Wigner distribution will form a link to such diverse fields as geometrical optics, ray optics, matrix optics, and radiometry.

Sections 1.2 through 1.7 will mainly deal with optical signals and systems. We treat the description of completely coherent and partially coherent light fields in Section 1.2. The Wigner distribution is introduced in Section 1.3 and elucidated with some optical examples. Properties of the Wigner distribution are considered in Section 1.4. In Section 1.5 we restrict ourselves to the one-dimensional case and observe the strong connection of the Wigner distribution to the fractional Fourier transformation and rotations in phase space. The propagation of the Wigner distribution through Luneburg’s first-order optical systems is the topic of Section 1.6, while the propagation of its moments is discussed in Section 1.7.
1.2 Elementary description of optical signals and systems

We consider scalar optical signals, which can be described by, say, \[ \tilde{f}(x, y, z, t) \], where \( x, y, z \) denote space variables and \( t \) represents the time variable. Very often we consider signals in a plane \( z = \text{constant} \), in which case we can omit the longitudinal space variable \( z \) from the formulas. Furthermore, the transverse space variables \( x \) and \( y \) are combined into a two-dimensional column vector \( \mathbf{r} \). The signals with which we are dealing are thus described by a function \[ \tilde{f}(\mathbf{r}, t) \].

Although real-world signals are real, we will not consider these signals as such. The signals \( \tilde{f}(\mathbf{r}, t) \) that we consider in this chapter are ‘analytic signals,’ and our real-world signals follow as the real part of these analytic signals.

We will throughout denote column vectors by bold-face, lower-case symbols, while matrices will be denoted by bold-face, upper-case symbols; transposition of vectors and matrices is denoted by the superscript \( ^t \). Hence, for instance, the two-dimensional column vectors \( \mathbf{r} \) and \( \mathbf{q} \) represent the space and spatial-frequency variables \( [x, y]^t \) and \( [u, v]^t \), respectively, and \( \mathbf{q}^t \mathbf{r} \) represents the inner product \( ux + vy \). Moreover, in integral expressions, \( d\mathbf{r} \) and \( d\mathbf{q} \) are shorthand notations for \( dx\,dy \) and \( du\,dv \), respectively.

1.2.1 Impulse response and coherent point-spread function

The input-output relationship of a general linear system \( \tilde{f}_i(r, t) \rightarrow \tilde{f}_o(r, t) \) reads

\[ \tilde{f}_o(\mathbf{r}_o, t_o) = \iint \tilde{h}(\mathbf{r}_o, \mathbf{r}_i, t_o, t_i) \tilde{f}_i(\mathbf{r}_i, t_i) \, d\mathbf{r}_i \, dt_i, \quad (1.1) \]

where \( \tilde{h}(\mathbf{r}_o, \mathbf{r}_i, t_o, t_i) \) is the impulse response, i.e., the system’s response to a Dirac function:

\[ \delta(\mathbf{r} - \mathbf{r}_i) \delta(t - t_i) \rightarrow \tilde{h}(\mathbf{r}_o, \mathbf{r}_i, t, t_i). \]

We will restrict ourselves to a time-invariant system, \( \tilde{h}(\mathbf{r}_o, \mathbf{r}_i, t_o, t_i) =: \tilde{h}(\mathbf{r}_o, \mathbf{r}_i, t_o - t_i) \), in which case the input-output relationship takes the form of a convolution (as far as the time variable is concerned):

\[ \tilde{f}_o(\mathbf{r}_o, t_o) = \iint \tilde{h}(\mathbf{r}_o, \mathbf{r}_i, t_o - t_i) \tilde{f}_i(\mathbf{r}_i, t_i) \, d\mathbf{r}_i \, dt_i. \quad (1.2) \]

The temporal Fourier transform of the impulse response \( \tilde{h}(\mathbf{r}_o, \mathbf{r}_i, \tau) \),

\[ h(\mathbf{r}_o, \mathbf{r}_i, \nu) = \int \tilde{h}(\mathbf{r}_o, \mathbf{r}_i, \tau) \exp(i2\pi\nu\tau) \, d\tau =: h(\mathbf{r}_o, \mathbf{r}_i), \quad (1.3) \]
is known as the coherent point-spread function; note that we will throughout omit the explicit expression of the temporal frequency $\nu$. If the temporal Fourier transform of the signal exists,

$$f(r, \nu) = \int \tilde{f}(r, t) \exp(i2\pi\nu t) \, dt =: f(r),$$  

(1.4)

we can formulate the input-output relationship in the temporal-frequency domain as

$$f_o(r_o) = \int h(r_o, r_i) f_i(r_i) \, dr_i.$$  

(1.5)

1.2.2 Mutual coherence function and cross-spectral density

How shall we proceed if the temporal Fourier transform of the signal does not exist? This happens in the general case of partially coherent light, where the signal $\tilde{f}(r, t)$ should be considered as a stochastic process. We then start with the mutual coherence function

$$\tilde{\Gamma}(r_1, r_2, t_1, t_2) = \mathbb{E}\{\tilde{f}(r_1, t_1) \tilde{f}^*(r_2, t_2)\} =: \tilde{\Gamma}(r_1, r_2, t_1 - t_2),$$  

(1.6)

where we have assumed that the stochastic process is temporally stationary. After Fourier transforming the mutual coherence function $\tilde{\Gamma}(r_1, r_2, \tau)$, we get the mutual power spectrum or cross-spectral density:

$$\Gamma(r_1, r_2, \nu) = \int \tilde{\Gamma}(r_1, r_2, \tau) \exp(i2\pi\nu \tau) \, d\tau =: \Gamma(r_1, r_2).$$  

(1.7)

The basic property of $\Gamma(r_1, r_2)$ is that it is a nonnegative definite Hermitian function of $r_1$ and $r_2$, i.e.,

$$\Gamma(r_1, r_2) = \Gamma^*(r_2, r_1) \quad \text{and} \quad \iint g(r_1) \Gamma(r_1, r_2) g^*(r_2) \, dr_1 \, dr_2 \geq 0$$  

(1.8)

for any function $g(r)$. The input-output relationship can now be formulated in the temporal-frequency domain as

$$\Gamma_o(r_1, r_2) = \iint h(r_1, \rho_1) \Gamma_i(\rho_1, \rho_2) h^*(\rho_2, r_2) \, d\rho_1 \, d\rho_2,$$  

(1.9)

which expression replaces Eq. (1.5). Note that in the completely coherent case, for which $\Gamma(r_1, r_2)$ takes the product form $f(r_1) f^*(r_2)$, the coherence is preserved and Eq. (1.9) reduces to Eq. (1.5).

1.2.3 Some basic examples of optical signals

Important basic examples of coherent signals, as they appear in a plane $z = \text{constant}$, are
(i). an impulse in that plane at position $r_0$, $f(r) = \delta(r - r_0)$. In optical terms, the impulse corresponds to a point source;
(ii). the crossing with that plane of a plane wave with spatial frequency $q_0$, $f(r) = \exp(i2\pi q_0^t r)$. The plane-wave example shows us how we should interpret the spatial-frequency vector $q_0$. We assume that the wavelength of the light equals $\lambda_0$, in which case the length of the wave vector $k$ equals $2\pi/\lambda_0$. If we express the wave vector in the form $k = [k_x, k_y, k_z]^t$, then $2\pi q_0 = 2\pi[q_x, q_y]^t = [k_x, k_y]^t$ is simply the transversal part of $k$, i.e., its projection onto the plane $z = \text{constant}$. Furthermore, if the angle between the wave vector $k$ and the $z$ axis equals $\theta$, then the length of the spatial-frequency vector $q_0$ equals $\sin \theta/\lambda_0$;
(iii). the crossing with that plane of a spherical wave (in the paraxial approximation), $f(r) = \exp(i\pi r^t H r)$, whose curvature is described by the real symmetric $2 \times 2$ matrix $H = H^t$. We use this example to introduce the ‘instantaneous’ frequency of a signal $|f(r)| \exp[i2\pi \phi(r)]$ as the derivative $d\phi/dr = \nabla \phi(r) = [\partial \phi/\partial x, \partial \phi/\partial y]^t$ of the signal’s argument. In the case of a spherical wave we have $d\phi/dr = H r$, and the instantaneous frequency corresponds to the normal on the spherical wave front.

Basic examples of partially coherent signals are
(iv). completely incoherent light with intensity $p(r)$, $\Gamma(r_1, r_2) = p(r_1) \delta(r_1 - r_2)$. Note that $p(r)$ is a nonnegative function;
(v). spatially stationary light, $\Gamma(r_1, r_2) = s(r_1 - r_2)$. We will see later that the Fourier transform of $s(r)$ is a nonnegative function.

1.3 Wigner distribution and ambiguity function
In this section we introduce the Wigner distribution and its Fourier transform, the ambiguity function.

1.3.1 Definitions
We introduce the spatial Fourier transforms of $f(r)$ and $\Gamma(r_1, r_2)$:

\[
\tilde{f}(q) = \int f(r) \exp(-i2\pi q^t r) \, dr,
\]
\[
\tilde{\Gamma}(q_1, q_2) = \int \int \Gamma(r_1, r_2) \exp(-i2\pi(q_1^t r_1 - q_2^t r_2)) \, dr_1 \, dr_2.
\]

We will throughout use the generic form $\Gamma(r_1, r_2)$, even in the case of completely coherent light, where we could use the product form $f(r_1)f^*(r_2)$. We thus elaborate on Eq. (1.11) and apply the coordinate transformation

\[
\begin{align*}
  r_1 &= r + \frac{1}{2}r', \\
  r_2 &= r - \frac{1}{2}r', \\
  r &= \frac{1}{2}(r_1 + r_2), \\
  r' &= r_2 - r_1,
\end{align*}
\]
Chapter 1. Wigner Distribution in Optics

and similarly for $q$. Note that the Jacobian equals one, so that $\text{d}r_1 \text{d}r_2 = \text{d}r \text{d}r'$. The Wigner distribution\(^1\) $W(r, q)$ and ambiguity function\(^18\) $A(r', q')$ now arise ‘midway’ between the cross-spectral density $\Gamma(r_1, r_2)$ and its Fourier transform $\bar{\Gamma}(q_1, q_2)$.

\[
\Gamma(q + \frac{1}{2}q', q - \frac{1}{2}q') = \iint \Gamma(r + \frac{1}{2}r', r - \frac{1}{2}r') \exp[-i2\pi(q'r' + r'q')] \, \text{d}r \, \text{d}r' \\
= \int W(r, q) \exp(-i2\pi r'q') \, \text{d}r = \int A(r', q') \exp(-i2\pi q'r') \, \text{d}r',
\]

and their definitions follow as

\[
W(r, q) = \int \Gamma(r + \frac{1}{2}r', r - \frac{1}{2}r') \exp(-i2\pi q'r') \, \text{d}r', \\
A(r', q') = \int \Gamma(r + \frac{1}{2}r', r - \frac{1}{2}r') \exp(-i2\pi q'r') \, \text{d}r,
\]

(1.13)

(1.14)

We immediately notice the realness of the Wigner distribution, and the Fourier transform relationship between the Wigner distribution and the ambiguity function:

\[
A(r', q') = \int W(r, q) \exp[-i2\pi(r'q' - q'r')] \, \text{d}r \, \text{d}q = \mathcal{F}[W(r, q)](r', q').
\]

(1.16)

This Fourier transform relationship implies that properties for the Wigner distribution have their counterparts for the ambiguity function and vice versa: moments for the Wigner distribution become derivatives for the ambiguity function, convolutions in the ‘Wigner domain’ become products in the ‘ambiguity domain,’ etc.

We like to present the cross-spectral density $\Gamma$, its spatial Fourier transform $\bar{\Gamma}$, the Wigner distribution $W$, and the ambiguity function $A$ at the corners of a rectangle, see Fig. 1.1 Along the sides of this rectangle we have Fourier transformations, $r' \rightarrow q$ and $r \rightarrow q'$, and their inverses, while along the diagonals we have double Fourier transformations, $(r, r') \rightarrow (q', q)$ and $(r, q) \rightarrow (q', r')$.

A distribution according to definitions (1.14) was introduced in optics by Dolin\(^2\) and Walther\(^3,4\) in the field of radiometry; Walther called it the generalized radiance. A few years later it was re-introduced, mainly in the field of Fourier optics.\(^5−11\) The ambiguity function was introduced in optics by Papoulis.\(^19\) The ambiguity function is treated in more detail in Chapter 2 by Jean-Pierre Guigay; in this chapter we concentrate on the Wigner distribution.
1.3. Wigner distribution and ambiguity function

\begin{equation}
\Gamma(r + \frac{1}{2}r', r - \frac{1}{2}r')
\end{equation}

\begin{equation}
W(r, q)
\end{equation}

\begin{equation}
\bar{\Gamma}(q + \frac{1}{2}q', q - \frac{1}{2}q')
\end{equation}

\begin{equation}
A(r', q')
\end{equation}

Figure 1.1 Schematic representation of the cross-spectral density \( \Gamma \), its spatial Fourier transform \( \bar{\Gamma} \), the Wigner distribution \( W \), and the ambiguity function \( A \), on a rectangle.

### 1.3.2 Some basic examples again

Let us return to our basic examples. The space behavior \( f(r) \) or \( \Gamma(r_1, r_2) \), the spatial-frequency behavior \( \hat{f}(q) \) or \( \hat{\Gamma}(q_1, q_2) \), and the Wigner distribution \( W(r, q) \) of (i) a point source, (ii) a plane wave, (iii) a spherical wave, (iv) an incoherent light field, and (v) a spatially-stationary light field, have been represented in Table 1.1.

We remark the clear physical interpretations of the Wigner distributions.

<table>
<thead>
<tr>
<th></th>
<th>( f(r) ) or ( \Gamma(r_1, r_2) )</th>
<th>( \hat{f}(q) ) or ( \hat{\Gamma}(q_1, q_2) )</th>
<th>( W(r, q) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>( \delta(r - r_0) )</td>
<td>( \exp(-i2\pi r'_0 q) )</td>
<td>( \delta(r - r_0) )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( \exp(i2\pi q'_1 r) )</td>
<td>( \delta(q - q_0) )</td>
<td>( \delta(q - q_0) )</td>
</tr>
<tr>
<td>(iii)</td>
<td>( \exp(i\pi r' H r) )</td>
<td>( [\det(-iH)]^{-1/2} \exp(-i\pi q' H^{-1} q) )</td>
<td>( \delta(q - H r) )</td>
</tr>
<tr>
<td>(iv)</td>
<td>( p(r_1) \delta(r_1 - r_2) )</td>
<td>( \bar{p}(q_1 - q_2) )</td>
<td>( p(r) )</td>
</tr>
<tr>
<td>(v)</td>
<td>( s(r_1 - r_2) )</td>
<td>( \bar{s}(q_1) \delta(q_1 - q_2) )</td>
<td>( \bar{s}(q) )</td>
</tr>
</tbody>
</table>

Table 1.1 Wigner distribution of some basic examples: (i) point source, (ii) plane wave, (iii) spherical wave, (iv) incoherent light, and (v) spatially-stationary light.

(i). The Wigner distribution of a point source \( f(r) = \delta(r - r_o) \) reads \( W(r, q) = \delta(r - r_o) \), and we observe that all the light originates from one point \( r = r_o \) and propagates uniformly in all directions \( q \).

(ii). Its dual, a plane wave \( f(r) = \exp(i2\pi q'_0 r) \), also expressible in the frequency domain as \( \hat{f}(q) = \delta(q - q_o) \), has as its Wigner distribution \( W(r, q) = \delta(q - q_o) \), and we observe that for all positions \( r \) the light propagates in only
one direction \( q_0 \).

(iii). The Wigner distribution of the spherical wave \( f(r) = \exp(i\pi r^t \mathbf{H} r) \) takes the simple form \( W(r, q) = \delta(q - \mathbf{H} r) \), and we conclude that at any point \( r \) only one frequency \( q = \mathbf{H} r \), the instantaneous frequency, manifests itself. This corresponds exactly to the ray picture of a spherical wave.

(iv). Incoherent light, \( \Gamma(r + \frac{1}{2} r', r - \frac{1}{2} r') = p(r') \delta(r') \), yields the Wigner distribution \( W(r, q) = p(r) \). Note that it is a function of the space variable \( r \) only, and that it does not depend on the frequency variable \( q \): the light radiates equally in all directions, with intensity profile \( p(r) \geq 0 \).

(v). Spatially stationary light, \( \Gamma(r + \frac{1}{2} r', r - \frac{1}{2} r') = s(r') \), is dual to incoherent light: its frequency behavior is similar to the space behavior of incoherent light and vice versa, and \( s(q) \), its intensity function in the frequency domain, is nonnegative. The duality between incoherent light and spatially stationary light is, in fact, the Van Cittert-Zernike theorem.

The Wigner distribution of spatially stationary light reads as \( W(r, q) = \bar{s}(q) \); note that it is a function of the frequency variable \( q \) only, and that it does not depend on the space variable \( r \). It thus has the same form as the Wigner distribution of incoherent light, except that it is rotated through 90° in the space-frequency domain. The same observation can be made for the point source and the plane wave, see examples (i) and (ii), which are also each other’s duals.

We illustrate the Wigner distribution of the one-dimensional spherical wave \( f(x) = \exp(i\pi hx^2) \), see example (iii) above, by a numerical simulation. To calculate \( W(x, u) \) practically, we have to restrict the integration interval for \( x' \). We model this by using a window function \( w(\frac{1}{2} x') \), so that the Wigner distribution takes the form

\[
P(x, u; w) = \int f(x + \frac{1}{2} x') w(\frac{1}{2} x') w^*(\frac{1}{2} x') f^*(x - \frac{1}{2} x') \exp(-i2\pi ux') \, dx'.
\]

(1.17)

The function \( P(x, u; w) \) is called the pseudo-Wigner distribution. It is common to choose an even window function, \( w(\frac{1}{2} x') = w(-\frac{1}{2} x') \), so that we have \( w(\frac{1}{2} x') w^*(-\frac{1}{2} x') = |w(\frac{1}{2} x')|^2 \). Fig. 1.2 shows the (pseudo) Wigner distribution of the signal \( f(x) = \exp(i\pi hx^2) \), which reads as

\[
\int |w(\frac{1}{2} x')|^2 \exp[-i2\pi(u - hx)] \, dx' = \mathcal{F} \left[ |w(\frac{1}{2} x')|^2 \right] (u - hx) \simeq \delta(u - hx),
\]

where we have chosen a rectangular window of width \( X \) in case (a),

\[
w(\frac{1}{2} x') = \text{rect}(x'/X),
\]

and a Hann(ing) window of width \( X \) in case (b),

\[
w(\frac{1}{2} x') = \cos^2(\pi x'/X) \text{ rect}(x'/X).
\]
Note the effect of $\mathcal{F}[|w(\frac{1}{2}x')|^2]$, which results in a sinc-type behavior in the case of the rectangular window, $P(x, u; w) = \sin[\pi(u - hx)X]/\pi(u - hx)$, and in a nonnegative but smoother version in the case of the Hann(ing) window.

1.3.3 Gaussian light

Gaussian light is an example that we will treat in some more detail. The cross-spectral density of the most general partially coherent Gaussian light can be written in the form

$$\Gamma(r_1, r_2) = 2\sqrt{\det G_1 \det G_2} \exp \left( -\frac{\pi}{2} \left[ \begin{array}{c} r_1 + r_2 \\ r_1 - r_2 \end{array} \right]^t \left[ \begin{array}{cc} G_1 & -iH \\ -iH^t & G_2 \end{array} \right] \left[ \begin{array}{c} r_1 + r_2 \\ r_1 - r_2 \end{array} \right] \right),$$

(1.18)

where we have chosen a representation that enables us to determine the Wigner distribution of such light in an easy way. The exponent shows a quadratic form in which a 4-dimensional column vector $[\begin{array}{c} (r_1 + r_2)^t \\ (r_1 - r_2)^t \end{array}]$ arises, together with a symmetric $4 \times 4$ matrix. This matrix consists of four real $2 \times 2$ submatrices $G_1, G_2, H,$ and $H^t$, where, moreover, the matrices $G_1$ and $G_2$ are positive definite symmetric. The special form of the matrix is a direct consequence of the fact that the cross-spectral density is a nonnegative definite Hermitian function. The Wigner distribution of such Gaussian light takes the form

$$W(r, q) = 4 \sqrt{\frac{\det G_1}{\det G_2}} \exp \left( -2\pi \left[ \begin{array}{c} r \\ q \end{array} \right]^t \left[ \begin{array}{cc} G_1 + H G_2^{-1} H^t & -H G_2^{-1} \\ -H^t G_2^{-1} & G_2^{-1} \end{array} \right] \left[ \begin{array}{c} r \\ q \end{array} \right] \right).$$

(1.19)

In a more common way, the cross-spectral density of general Gaussian light
(with ten degrees of freedom) can be expressed in the form

\[
\Gamma(r_1, r_2) = 2\sqrt{\det G_1} \exp\left\{ -\frac{1}{2}\pi (r_1 - r_2)^t G_0 (r_1 - r_2) \right\} \\
\times \exp\{-\pi r_1^t [G_1 - i\frac{1}{2}(H + H')] r_1\} \exp\{-\pi r_2^t [G_1 + i\frac{1}{2}(H + H')] r_2\} \\
\times \exp\{-\pi i r_1^t (H - H') r_2\}, \quad (1.20)
\]

where we have introduced the real, positive definite symmetric \(2 \times 2\) matrix \(G_0 = G_2 - G_1\). Note that the asymmetry of the matrix \(H\) is a measure for the twist of Gaussian light, and that general Gaussian light reduces to zero-twist Gaussian Schell-model light,\(^{27, 28}\) if the matrix \(H\) is symmetric, \(H - H' = 0\). In that case, the light can be considered as spatially stationary light with a Gaussian cross-spectral density \(2\sqrt{\det G_1} \exp\left\{ -\frac{1}{2}\pi (r_1 - r_2)^t G_0 (r_1 - r_2) \right\}\), modulated by a Gaussian modulator with modulation function \(\exp\{-\pi r^t (G_1 - iH) r\}\). We remark that such Gaussian Schell-model light (with nine degrees of freedom) forms a large subclass of Gaussian light; it applies, for instance, in

- the completely coherent case \((H = H', G_0 = 0, G_1 = G_2)\),
- the (partially coherent) one-dimensional case \((g_0 = g_2 - g_1 \geq 0)\), and
- the (partially coherent) rotationally symmetric case \((H = hI, G_1 = g_1 I, G_2 = g_2 I, G_0 = (g_2 - g_1)I)\), with \(I\) the \(2 \times 2\) identity matrix.

Gaussian Schell-model light reduces to so-called symplectic Gaussian light,\(^{21}\) if the matrices \(G_0, G_1, G_2\) are proportional to one another, \(G_1 = \sigma G, G_2 = \sigma^{-1} G, G_0 = (\sigma^{-1} - \sigma)G\), with \(G\) a real, positive definite symmetric \(2 \times 2\) matrix and \(0 < \sigma \leq 1\). The Wigner distribution then takes the form

\[
W(r, q) = 4\sigma^2 \exp\left( -2\pi\sigma \begin{bmatrix} r \\ q \end{bmatrix}^t \begin{bmatrix} G + HG^{-1}H & -HG^{-1}G^{-1} \\ -G^{-1}H & G^{-1} \end{bmatrix} \begin{bmatrix} r \\ q \end{bmatrix} \right). \quad (1.21)
\]

The name symplectic Gaussian light (with six degrees of freedom) originates from the fact that the \(4 \times 4\) matrix that arises in the exponent of the Wigner distribution (1.21) is symplectic. We will return to symplecticity later in this chapter. We remark that symplectic Gaussian light forms a large subclass of Gaussian Schell-model light; it applies again, for instance, in the completely coherent case, in the (partially coherent) one-dimensional case, and in the (partially coherent) rotationally symmetric case. And again: symplectic Gaussian light can be considered as spatially stationary light with a Gaussian cross-spectral density, modulated by a Gaussian modulator, cf. Eq. (1.20), but now with the real parts of the quadratic forms in the two exponents described – up to a positive constant – by the same real, positive definite symmetric matrix \(G\).
1.3.4 Local frequency spectrum

The Wigner distribution can be considered as a local frequency spectrum; the marginals are correct

\[
\Gamma(r, r) = \int W(r, q) \, dq \quad \text{and} \quad \bar{\Gamma}(q, q) = \int W(r, q) \, dr. \tag{1.22}
\]

Integrating over all frequency values \(q\) yields the intensity \(\Gamma(r, r)\) of the signal’s representation in the space domain, and integrating over all space values \(r\) yields the intensity \(\bar{\Gamma}(q, q)\) of the signal’s representation in the frequency domain. To operate easily in the mixed \(rq\) plane, the so-called phase space, we will benefit from normalization to dimensionless coordinates, \(W^{-1}r =: r\) and \(Wq =: q\), where \(W\) is a diagonal matrix with positive diagonal entries

\[
W = \begin{bmatrix}
w_x & 0 \\
0 & w_y
\end{bmatrix}. \tag{1.23}
\]

In subsequent sections, we will often work with these normalized coordinates; it will be clear from the context whether or not normalization is necessary.

1.4 Some properties of the Wigner distribution

Let us consider some of the important properties of the Wigner distribution. We consider in particular properties that are specific for partially coherent light. Additional properties of the Wigner distribution, especially of the Wigner distribution in the completely coherent case, can be found elsewhere; see, for instance, Refs. 29–40 and the many references cited therein.

1.4.1 Inversion formula

The definition (1.14) of the Wigner distribution \(W(r, q)\) has the form of a Fourier transformation of the cross-spectral density \(\Gamma(r + \frac{1}{2}r', r - \frac{1}{2}r')\) with \(r'\) and \(q\) as conjugated variables and with \(r\) as a parameter. The cross-spectral density can thus be reconstructed from the Wigner distribution simply by applying an inverse Fourier transformation.

1.4.2 Shift covariance

The Wigner distribution satisfies the important property of space and frequency shift covariance: if \(W(r, q)\) is the Wigner distribution that corresponds to \(\Gamma(r_1, r_2)\), then \(W(r - r_0, q - q_0)\) is the Wigner distribution that corresponds to the space and frequency shifted version \(\Gamma(r_1 - r_0, r_2 - r_0) \exp[\text{i}2\pi q_0^t(r_1 - r_2)]\).

1.4.3 Radiometric quantities

Although the Wigner distribution is real, it is not necessarily nonnegative; this prohibits a direct interpretation of the Wigner distribution as an energy density
function (or radiance function). Friberg has shown\textsuperscript{41} that it is not possible to define a radiance function that satisfies all the physical requirements from radiometry; in particular, as we mentioned, the Wigner distribution has the physically unattractive property that it may take negative values.

Nevertheless, several integrals of the Wigner distribution have clear physical meanings and can be interpreted as radiometric quantities. We mentioned already that the integral over the frequency variable, \( \int W(\mathbf{r}, \mathbf{q}) \, d\mathbf{q} = \Gamma(\mathbf{r}, \mathbf{r}) \), represents the intensity of the signal, whereas the integral over the space variable, \( \int W(\mathbf{r}, \mathbf{q}) \, d\mathbf{r} = \bar{\Gamma}(\mathbf{q}, \mathbf{q}) \), yields the intensity of the signal’s Fourier transform; the latter is, apart from the usual factor \( \cos^2 \theta \) (where \( \theta \) is the angle of observation with respect to the \( z \)-axis), proportional to the radiant intensity.\textsuperscript{42, 43} The total energy \( E \) of the signal follows from the integral over the entire space-frequency domain:

\[
E = \iint W(\mathbf{r}, \mathbf{q}) \, d\mathbf{r} \, d\mathbf{q}.
\] (1.24)

The real symmetric \( 4 \times 4 \) matrix \( \mathbf{M} \) of normalized second-order moments, defined by

\[
\begin{align*}
\mathbf{M} &= \frac{1}{E} \iint \begin{bmatrix} \mathbf{r} \\ \mathbf{q} \end{bmatrix} \begin{bmatrix} \mathbf{r}' \\ \mathbf{q}' \end{bmatrix} W(\mathbf{r}, \mathbf{q}) \, d\mathbf{r} \, d\mathbf{q} = \frac{1}{E} \iint \begin{bmatrix} \mathbf{r}' & \mathbf{r} \\ \mathbf{q}' & \mathbf{q} \end{bmatrix} W(\mathbf{r}, \mathbf{q}) \, d\mathbf{r} \, d\mathbf{q} \\
&= \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} & m_{xv} \\ m_{yx} & m_{yy} & m_{yz} & m_{yv} \\ m_{zx} & m_{zy} & m_{zz} & m_{zv} \\ m_{vx} & m_{vy} & m_{vz} & m_{vv} \end{bmatrix},
\end{align*}
\] (1.25)

yields such quantities as the effective width \( d_x = \sqrt{m_{xx}} \) of the intensity \( \Gamma(\mathbf{r}, \mathbf{r}) \) in the \( x \)-direction

\[
m_{xx} = \frac{1}{E} \iint x^2 W(\mathbf{r}, \mathbf{q}) \, d\mathbf{r} \, d\mathbf{q} = \frac{1}{E} \iint x^2 \Gamma(\mathbf{r}, \mathbf{r}) \, d\mathbf{r} = d_x^2
\] (1.26)

and, similarly, the effective width \( d_u = \sqrt{m_{uu}} \) of the intensity \( \bar{\Gamma}(\mathbf{q}, \mathbf{q}) \) in the \( u \)-direction, but it also yields all kinds of mixed moments. It will be clear that the main-diagonal entries of the moment matrix \( \mathbf{M} \), being interpretable as squares of effective widths, are positive. As a matter of fact, it can be shown that the matrix \( \mathbf{M} \) is positive definite; see, for instance, Refs. 44–46.

The radiant emittance\textsuperscript{42, 43} is equal to the integral

\[
j_z(\mathbf{r}) = \int \frac{\sqrt{k^2 - (2\pi)^2 \mathbf{q}' \mathbf{q}}}{k} W(\mathbf{r}, \mathbf{q}) \, d\mathbf{q}
\] (1.27)

where \( k = 2\pi/\lambda_0 \) represents the usual wave number. When we combine the radiant emittance \( j_z \) with the two-dimensional vector

\[
j_r(\mathbf{r}) = \int \frac{2\pi \mathbf{q}}{k} W(\mathbf{r}, \mathbf{q}) \, d\mathbf{q},
\] (1.28)
we can construct the three-dimensional vector \([j_x, j_y, j_z]\), which is known as the geometrical vector flux.\(^{47}\) The total radiant flux \(\int j_z(r) \, dr\) follows from integrating the radiant emittance over the space variable \(r\). More on radiometry can be found in Chapter 7 by Arvind Marathay.

### 1.4.4 Instantaneous frequency

The Wigner distribution \(W_f(r, q)\) satisfies the nice property that for a coherent signal \(f(r) = |f(r)| \exp[i2\pi\phi(r)]\), the instantaneous frequency \(d\phi/dr = \nabla \phi(r)\) follows from \(W_f(r, q)\) through

\[
\frac{d\phi}{dr} = \frac{\int q W_f(r, q) \, dq}{\int W_f(r, q) \, dq}. \tag{1.29}
\]

To prove this property, we proceed as follows. From \(f(r) = |f(r)| \exp[i2\pi\phi(r)]\), we get

\[
\ln f(r) = \ln |f(r)| + i2\pi\phi(r),
\]

hence \(\text{Im}\{\ln f(r)\} = 2\pi\phi(r)\), which then leads to the identity

\[
2\pi \frac{d\phi(r)}{dr} = \text{Im} \left\{ \frac{d \ln f(r)}{d r} \right\} = \text{Im} \left\{ \frac{\nabla f(r)}{f(r)} \right\}
= \frac{1}{2i} \left[ \frac{\nabla f(r)}{f(r)} - \frac{\left( \frac{\nabla f(r)}{f(r)} \right)^*}{f(r)} \right] = \frac{1}{2i} \frac{\nabla f(r) [f^*(r) - f(r)] [\nabla f(r)]^*}{f(r) f^*(r)}
= -i \frac{1}{|f(r)|^2} \frac{\partial}{\partial r'} \left[ f(r + \frac{1}{2} r') f^*(r - \frac{1}{2} r') \right] \bigg|_{r' = 0}.
\]

On the other hand we have the identity

\[
2\pi \int q W_f(r, q) \, dq
= 2\pi \int \left[ \int f(r + \frac{1}{2} r') f^*(r - \frac{1}{2} r') \exp(-i2\pi q' r') \, dr' \right] q \, dq
= \int f(r + \frac{1}{2} r') f^*(r - \frac{1}{2} r') \left[ 2\pi \int q \exp(-i2\pi q' r') \, dq \right] \, dr'
= i \int f(r + \frac{1}{2} r') f^*(r - \frac{1}{2} r') [\nabla \delta(r')] \, dr'
= -i \frac{\partial}{\partial r'} \left[ f(r + \frac{1}{2} r') f^*(r - \frac{1}{2} r') \right] \bigg|_{r' = 0},
\]

and when we combine these two results, we immediately get Eq. (1.29). It is this property in particular that made the Wigner distribution a popular tool for the determination of the instantaneous frequency.
1.4.5 Moyal’s relationship

An important relationship between the Wigner distributions of two signals and the cross-spectral densities of these signals, which is an extension to partially coherent light of a relationship formulated by Moyal\textsuperscript{48} for completely coherent light, reads as

$$\int \int W_1(r, q) W_2(r, q) \, dr \, dq = \int \int \Gamma_1(r_1, r_2) \Gamma_2^*(r_1, r_2) \, dr_1 \, dr_2,$$

and

$$\int \int \bar{\Gamma}_1(q_1, q_2) \bar{\Gamma}_2^*(q_1, q_2) \, dq_1 \, dq_2.$$

(1.30)

This relationship has an application in averaging one Wigner distribution with another one, which averaging always yields a nonnegative result.

1.5 One-dimensional case and the fractional Fourier transformation

Let us for the moment restrict ourselves to coherent light and to the one-dimensional case, and let us use normalized coordinates. The signal is now written as $f(x)$.

1.5.1 Fractional Fourier transformation

An important transformation with respect to operations in a phase space, is the fractional Fourier transformation, which reads as\textsuperscript{49–53}

$$f_\gamma(x_o) = F_\gamma(x_o) = \exp\left(\frac{i \gamma}{2} \right) \int \exp\left[ \frac{i \pi}{\sin \gamma} \left( x_i^2 + x_o^2 \cos \gamma - 2 x_o x_i \right) \sin \gamma \right] f_i(x_i) \, dx_i \quad (\gamma \neq n\pi),$$

(1.31)

where $\sqrt{i \sin \gamma}$ is defined as $|\sin \gamma| \exp[i(\frac{1}{2} \pi) \text{sgn}(\sin \gamma)]$. We mention the special cases $F_0(x) = f(x)$, $F_{\pi/2}(x) = f(-x)$, and the common Fourier transform $F_{\pi/2}(x) = f(x)$. Two realizations of an optical fractional Fourier transformer have been proposed by Lohmann,\textsuperscript{50} see Fig. 1.3. For both cases we have $\sin^2(\frac{1}{2} \gamma) = d/2f$; the normalization width $w$ is related to the distance $d$ and the focal length of the lens $f$ by $w^2 \tan(\frac{1}{2} \gamma) = \lambda_o d$ for case (a) and by $w^2 \sin \gamma = \lambda_o d$ for case (b).

1.5.2 Rotation in phase space

In terms of the ray transformation matrix, which will be introduced and treated in more detail in Section 1.6, the fractional Fourier transformer is represented by

$$\begin{bmatrix} x_o \\ u_o \end{bmatrix} = \begin{bmatrix} w & 0 \\ 0 & w^{-1} \end{bmatrix} \begin{bmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{bmatrix} \begin{bmatrix} w^{-1} & 0 \\ 0 & w \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix}$$

(1.32)
and after normalization, \( w^{-1}x =: x \) and \( wu =: u \), we have the form
\[
\begin{bmatrix}
x_o \\
u_o
\end{bmatrix} = \begin{bmatrix}
\cos \gamma & \sin \gamma \\
-\sin \gamma & \cos \gamma
\end{bmatrix} \begin{bmatrix}
x_i \\
u_i
\end{bmatrix}.
\] (1.33)

The input-output relation of a fractional Fourier transformer in terms of the Wigner distribution is remarkably simple; if \( W_f \) denotes the Wigner distribution of \( f(x) \) and \( W_{F_{\gamma}} \) that of \( F_{\gamma}(x) \), we have
\[
W_{F_{\gamma}}(x, u) = W_f(x \cos \gamma - u \sin \gamma, x \sin \gamma + u \cos \gamma)
\] (1.34)
and we conclude that a fractional Fourier transformation corresponds to a rotation in phase space.

### 1.5.3 Generalized marginals – Radon transform

On the analogy of the two special cases \(|f(x)|^2 = \int W_f(x, u) \, du\) and \(|\tilde{f}(u)|^2 = \int W_f(x, u) \, dx\), which correspond to projections along the \( u \) and the \( x \) axis, respectively, we can now get an easy expression for the projection along an axis that is tilted through an angle \( \gamma \)
\[
|F_{\gamma}(x)|^2 = \int W_{F_{\gamma}}(x, u) \, du
= \int W_f(x \cos \gamma - u \sin \gamma, x \sin \gamma + u \cos \gamma) \, du
= \iint W_f(\xi, u) \delta(\xi \cos \gamma + u \sin \gamma - x) \, d\xi \, du.
\] (1.35)

We thus conclude that not only the marginals for \( \gamma = 0 \) and \( \gamma = \frac{1}{2} \pi \) are correct, but in fact any marginal for an arbitrary angle \( \gamma \). We observe a strong connection between the Wigner distribution \( W_f(x, u) \) and the intensity \(|F_{\gamma}(x)|^2\) of the signal’s fractional Fourier transform. Note also the relation to the Radon transform.

Since the ambiguity function is the two-dimensional Fourier transform of the Wigner distribution, we could also represent \(|F_{\gamma}(x)|^2\) in the form
\[
|F_{\gamma}(x)|^2 = \int A_{F_{\gamma}}(\rho \sin \gamma, -\rho \cos \gamma) \exp(-i2\pi x\rho) \, d\rho
\] (1.36)
and we conclude that the values of the ambiguity function along the line defined by the angle $\gamma$ and the projections of the Wigner distribution for the same angle $\gamma$ are related to each other by a Fourier transformation. Note that the ambiguity function in Eq. (1.36) is represented in a quasi-polar coordinate system $(\rho, \gamma)$.

We recall that the signal $f(x) = |f(x)| \exp[i2\pi\phi(x)]$ can be reconstructed by using the intensity profiles of the fractional Fourier transform $F_{\gamma}(x)$ for two close values of the fractional angle $\gamma$. The reconstruction procedure is based on the property

$$\frac{\partial |F_{\gamma}(x)|^2}{\partial \gamma} \bigg|_{\gamma=0} = -\frac{d}{dx} \left[ |f(x)|^2 \frac{d\phi(x)}{dx} \right], \quad (1.37)$$

which can be proved by first differentiating Eq. (1.35) with respect to $\gamma$ and using the identity

$$\frac{\partial}{\partial \gamma} \delta(\xi \cos \gamma + u \sin \gamma - x) \bigg|_{\gamma=0} = (-\xi \sin \gamma + u \cos \gamma) \delta'(\xi \cos \gamma + u \sin \gamma - x) \bigg|_{\gamma=0} = u \delta'((\xi - x),$$

leading to

$$\frac{\partial |F_{\gamma}(x)|^2}{\partial \gamma} \bigg|_{\gamma=0} = \iint u W_f(\xi, u) \delta'(\xi - x) \, d\xi \, du = -\frac{d}{dx} \left[ \int u W_f(x, u) \, du \right],$$

and then substituting from Eq. (1.29), $\int u W_f(x, u) \, du = |f(x)|^2 \frac{d\phi(x)}{dx}$. By measuring two intensity profiles around $\gamma = 0$, $|F_{\gamma_0}(x)|^2$ and $|F_{-\gamma_0}(x)|^2$ for instance, approximating $\partial |F_{\gamma}(x)|^2/\partial \gamma$ by $\left(|F_{\gamma_0}(x)|^2 - |F_{-\gamma_0}(x)|^2\right)/2\gamma_0$, and integrating the result, we get $|f(x)|^2 \frac{d\phi(x)}{dx}$. After dividing this by the intensity $|f(x)|^2 = |F_0(x)|^2$, which can be approximated by $\left(|F_{\gamma_0}(x)|^2 + |F_{-\gamma_0}(x)|^2\right)/2$, we find an approximation for the phase derivative $d\phi(x)/dx$, which after a second integration yields the phase $\phi(x)$. Together with the modulus $|f(x)|$, the signal $f(x)$ can thus be reconstructed. This procedure can be extended to other members of the class of Luneburg’s first-order optical systems, to be considered in the next section, in particular by using a section of free space instead of a fractional Fourier transformer.

### 1.6 Propagation of the Wigner distribution

In this section, we study how the Wigner distribution propagates through linear optical systems. We therefore consider an optical system as a black box, with an input plane and an output plane, and focus on the important class of first-order optical systems. A continuous medium, in which the signal must satisfy a certain differential equation, is considered in Section 1.6.5, but without going into much detail.
1.6. Propagation of the Wigner distribution

1.6.1 First-order optical systems – ray transformation matrix

An important class of optical systems is the class of Luneburg’s first-order optical systems. This class consists of a section of free space (in the Fresnel approximation), a thin lens, and all possible combinations of these. A first-order optical system can most easily be described in terms of its (normalized) ray transformation matrix

\[
\begin{bmatrix}
  r_o \\
  q_o \\
\end{bmatrix} = \begin{bmatrix}
  W & 0 \\
  0 & W^{-1} \\
\end{bmatrix} \begin{bmatrix}
  A & B \\
  C & D \\
\end{bmatrix} \begin{bmatrix}
  W^{-1} & 0 \\
  0 & W \\
\end{bmatrix} \begin{bmatrix}
  r_i \\
  q_i \\
\end{bmatrix},
\]

(1.38)

which relates the position \( r_i \) and direction \( q_i \) of an incoming ray to the position \( r_o \) and direction \( q_o \) of the outgoing ray. In normalized coordinates, \( W^{-1}r =: r \) and \( Wq =: q \), we have

\[
\begin{bmatrix}
  r_o \\
  q_o \\
\end{bmatrix} = \begin{bmatrix}
  A & B \\
  C & D \\
\end{bmatrix} \begin{bmatrix}
  r_i \\
  q_i \\
\end{bmatrix}.
\]

(1.39)

We recall that the ray transformation matrix is symplectic. Using the matrix \( J \),

\[
J = i \begin{bmatrix}
  0 & -I \\
  I & 0 \\
\end{bmatrix} = J^{-1} = J^t = -J^t,
\]

(1.40)

where \( J^{-1}, J^t = (J^*)^t \), and \( J^t \) are the inverse, the adjoint, and the transpose of \( J \), respectively, symplecticity can be elegantly expressed as \( T^{-1} = JT^t J \). In detail we have

\[
T^{-1} = \begin{bmatrix}
  A & B \\
  C & D \\
\end{bmatrix}^{-1} = \begin{bmatrix}
  D^t & -B^t \\
  -C^t & A^t \\
\end{bmatrix} = J T^t J.
\]

(1.41)

If \( \det B \neq 0 \), the coherent point-spread function of the first-order optical system reads

\[
h(r_o, r_i) = (\det iB)^{-1/2} \exp[i\pi(r_o DB^{-1} r_o - 2r^t_i B^{-1} r_o + r^t_i B^{-1} A r_i)],
\]

(1.42)

see also Refs. 60 and 61. In the limiting case that \( B \to 0 \), we have

\[
h(r_o, r_i) = |\det A|^{-1/2} \exp(i\pi r^t_o CA^{-1} r_o) \delta(r_i - A^{-1} r_o).
\]

(1.43)

In the degenerate case \( \det B = 0 \) but \( B \neq 0 \), a representation in terms of the coherent point-spread function can also be formulated. The relationship between the input Wigner distribution \( W_i(r, q) \) and the output Wigner distribution \( W_o(r, q) \) takes the simple form

\[
W_o(A r + B q, C r + D q) = W_i(r, q),
\]

(1.44)

and this is independent of the possible degeneracy of the submatrix \( B \).
1.6.2 Phase-space rotators – more rotations in phase space

If the ray transformation matrix is not only symplectic but also orthogonal, $T^{-1} = T^\dagger$, the system acts as a general phase-space rotator,\(^{53}\) as we will see shortly. We then have $A = D$ and $B = -C$, and $U = A + iB$ is a unitary matrix: $U^\dagger = U^{-1}$. We thus have

$$T = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \quad \text{and} \quad (A - iB)^\dagger = U^\dagger = U^{-1} = (A + iB)^{-1}, \quad (1.45)$$

and hence

$$W_o(Ar + Bq, -Br + Aq) = W_i(r, q). \quad (1.46)$$

In the one-dimensional case, such a system reduces to a fractional Fourier transformer ($A = \cos \gamma, B = \sin \gamma$); the extension to a higher-dimensional separable fractional Fourier transformer (with diagonal matrices $A$ and $B$, and different fractional angles for the different coordinates) is straightforward.

In the two-dimensional case, the three basic systems with an orthogonal ray transformation matrix are (i) the separable fractional Fourier transformer $\mathcal{F}(\gamma_x, \gamma_y)$, (ii) the rotator $\mathcal{R}(\varphi)$, and (iii) the gyrator $\mathcal{G}(\varphi)$, with unitary representations $U = A + iB$ equal to

$$\begin{bmatrix} \exp(i\gamma_x) & 0 \\ 0 & \exp(i\gamma_y) \end{bmatrix}, \quad \begin{bmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \cos \varphi & i \sin \varphi \\ i \sin \varphi & \cos \varphi \end{bmatrix}, \quad (1.47)$$

respectively. All three systems correspond to rotations in phase space, which justifies the name phase-space rotators!

From the many decompositions of a general phase-space rotator into the more basic ones, we mention $\mathcal{F}(\frac{1}{2}, \gamma, -\frac{1}{2}, -\gamma) \mathcal{R}(\varphi) \mathcal{F}(\frac{1}{2}, -\gamma, \frac{1}{2}, \gamma) \mathcal{F}(\gamma_x, \gamma_y)$, which follows directly if we represent the unitary matrix as

$$U = \begin{bmatrix} \exp(i\gamma_x) \cos \varphi & \exp[i(\gamma_y + \gamma)] \sin \varphi \\ -\exp[i(\gamma_x - \gamma)] \sin \varphi & \exp(i\gamma_y) \cos \varphi \end{bmatrix}. \quad (1.48)$$

Note that we have the relationship $\mathcal{F}(\frac{1}{4}, \pi, -\frac{1}{4}, \pi) \mathcal{R}(\varphi) \mathcal{F}(\frac{1}{4}, \pi, -\frac{1}{4}, \pi) = \mathcal{G}(\varphi)$, which is just one of the many similarity-type relationships that exist between a rotator $\mathcal{R}(\alpha)$, a gyrator $\mathcal{G}(\beta)$, and an antisymmetric fractional Fourier transformer $\mathcal{F}(\gamma, -\gamma)$:

$$\mathcal{F}(\pm \frac{1}{4}, \pi, \pm \frac{1}{4}, \pi) \mathcal{G}(\pm \varphi) \mathcal{F}(\pm \frac{1}{4}, \pi, \pm \frac{1}{4}, \pi) = \mathcal{R}(\mp \varphi), \quad (1.49a)$$

$$\mathcal{F}(\pm \frac{1}{4}, \pi, \pm \frac{1}{4}, \pi) \mathcal{R}(\pm \varphi) \mathcal{F}(\mp \frac{1}{4}, \pi, \mp \frac{1}{4}, \pi) = \mathcal{G}(\varphi), \quad (1.49b)$$

$$\mathcal{R}(\pm \frac{1}{4}, \pi) \mathcal{F}(\pm \varphi, \mp \varphi) \mathcal{R}(\mp \frac{1}{4}, \pi) = \mathcal{G}(\varphi), \quad (1.49c)$$

$$\mathcal{G}(\pm \frac{1}{4}, \pi) \mathcal{F}(\pm \varphi, \mp \varphi) \mathcal{G}(\mp \frac{1}{4}, \pi) = \mathcal{R}(\varphi), \quad (1.49d)$$

$$\mathcal{R}(\mp \frac{1}{4}, \pi) \mathcal{G}(\pm \varphi) \mathcal{R}(\mp \frac{1}{4}, \pi) = \mathcal{F}(\varphi, -\varphi), \quad (1.49e)$$

$$\mathcal{G}(\pm \frac{1}{4}, \pi) \mathcal{R}(\pm \varphi) \mathcal{G}(\mp \frac{1}{4}, \pi) = \mathcal{F}(\varphi, \varphi). \quad (1.49f)$$
If we separate from $U$ the scalar matrix $U_f(\vartheta, \vartheta) = \exp(i\vartheta) I$ with $\exp(2i\vartheta) = \det U$, which matrix corresponds to a symmetric fractional Fourier transformer $\mathcal{F}(\vartheta, \vartheta)$, the remaining matrix is a so-called quaternion, and thus a $2 \times 2$ unitary matrix with unit determinant; expressed in the form of Eq. (1.48), this would mean $\gamma_y = -\gamma_x$. Note that the matrices $U_r(\alpha)$, $U_g(\beta)$, and $U_f(\gamma, -\gamma)$, corresponding to a rotator $R(\alpha)$, a gyrator $G(\beta)$, and an antisymmetric fractional Fourier transformer $\mathcal{F}(\gamma, -\gamma)$, respectively, are quaternions, and that every separable fractional Fourier transformer $\mathcal{F}(\gamma_x, \gamma_y)$ can be decomposed as $\mathcal{F}(\vartheta, \vartheta) \mathcal{F}(\gamma, -\gamma)$.

We easily verify – for instance by expressing the unitary matrix $U$ in the form of Eq. (1.48) – that the input-output relation for a phase-space rotator can be expressed in the form

$$r_o - iq_o = U (r_i - iq_i),$$

(1.50)

which is an easy alternative for Eq. (1.39). Phase-space rotators are considered in more details in Chapter 3 by Tatiana Alieva.

### 1.6.3 More general systems – ray-spread function

First-order optical systems are a perfect match for the Wigner distribution, since their point-spread function is a quadratic-phase function. Nevertheless, an input-output relationship can always be formulated for the Wigner distribution. In the most general case, based on the relationships (1.5) and (1.9), we write

$$W_o(r_o, q_o) = \iint K(r_o, q_o, r_i, q_i) W_i(r_i, q_i) \, dr_i \, dq_i$$

(1.51)

with

$$K(r_o, q_o, r_i, q_i) = \iint h(r_o + \frac{1}{2}r_o', r_i + \frac{1}{2}r_i') h^*(r_o - \frac{1}{2}r_o', r_i - \frac{1}{2}r_i')$$

$$\times \exp[-i2\pi(q_o r_o' - q_i r_i')] \, dr_o' \, dr_i'.$$

(1.52)

Relation (1.52) can be considered as the definition of a double Wigner distribution; hence, the function $K$ has all the properties of a Wigner distribution, for instance the property of realness.

Let us think about the physical meaning of the function $K$. In a formal way, the function $K$ is the response of the system in the space-frequency domain when the input signal is described by a product of two Dirac functions $W_i(r, q) = \delta(r - r_i) \delta(q - q_i)$; only in a formal way, since an actual input signal yielding such a Wigner distribution does not exist. Nevertheless, such an input signal could be considered as a single ray entering the system at the position $r_i$ with direction $q_i$. Hence, the function $K$ might be called the ray-spread function of the system.

### 1.6.4 Geometric-optical systems

Let us start by studying a modulator, described – in the case of partially coherent light – by the input-output relationship $\Gamma_o(r_1, r_2) = m(r_1) \Gamma_i(r_1, r_2) m^*(r_2)$. The
input and output Wigner distributions are related by the relationship

\[ W_o(r, q) = \int W_m(r, q - q_i) W_i(r, q_i) \, dq_i, \]  

(1.53)

where \( W_m(r, q) \) is the Wigner distribution of the modulation function \( m(r) \).

We now confine ourselves to the case of a pure phase modulation function \( m(r) = \exp\{i2\pi\phi(r)\} \). We then get

\[ m(r + \frac{1}{2}r') m^*(r - \frac{1}{2}r') = \exp\{i2\pi[\phi(r + \frac{1}{2}r') - \phi(r - \frac{1}{2}r')]\} \]

\[ = \exp\{i2\pi[(d\phi/dr)'r' + \text{higher-order terms}]\}. \]  

(1.54)

If we consider only the first-order derivative in relation (1.54), we get

\[ W_m(r, q) \approx \delta(q - d\phi/dr), \]

and the input-output relationship of the pure phase modulator becomes \( W_o(r, q) \approx W_i(r, q - d\phi/dr) \), which is a mere coordinate transformation. We conclude that a single input ray yields a single output ray.

The ideas described above have been applied to the design of optical coordinate transformerson\(^63, 64\) and to the theory of aberrations\(^65\). Now, if the first-order approximation is not sufficiently accurate, i.e., if we have to take into account higher-order derivatives in relation (1.54), the Wigner distribution allows us to overcome this problem. Indeed, we still have the exact input-output relationship (1.53) and we can take into account as many derivatives in relation (1.54) as necessary. We thus end up with a more general form\(^66\) than

\[ W_o(r, q) \approx W_i(r, q - d\phi/dr). \]

This will yield an Airy function instead of a Dirac function, for instance, when we take not only the first but also the third derivative into account.

We concluded that a single input ray yields a single output ray. This may also happen in more general – not just modulation-type – systems; we call such systems geometric-optical systems. These systems have the simple input-output relationship \( W_o(r, q) \approx W_i[g_x(r, q), g_u(r, q)] \), where the \( \approx \) sign becomes an \( = \) sign in the case of linear functions \( g_x \) and \( g_u \), i.e., in the case of Luneburg’s first-order optical systems. There appears to be a close relationship to the description of such geometric-optical systems by means of the Hamilton characteristics.\(^6\)

1.6.5 Transport equations

With the tools of this section, we could study the propagation of the Wigner distribution through free space by considering a section of free space as an optical system with an input plane and an output plane. It is possible, however, to find the propagation of the Wigner distribution through free space directly from the differential equation that the signal must satisfy. We therefore let the longitudinal variable \( z \) enter into the formulas and remark that the propagation of coherent light in free space (at least in the Fresnel approximation) is governed by the differential equation (see, for instance, Ref. 15, p. 358)

\[ -i\frac{\partial f}{\partial z} = \left( k + \frac{1}{2k} \frac{\partial^2}{\partial r^2} \right) f, \]  

(1.55)
with \( \partial^2 / \partial r^2 \) representing the scalar operator \( \partial^2 / \partial x^2 + \partial^2 / \partial y^2 \) and with \( k \) the wave number. The propagation of the Wigner distribution is now described by a so-called transport equation\(^7,\ 8,\ 67–70\) which in this case takes the form

\[
\frac{2\pi q_t}{k} \frac{\partial W}{\partial r} + \frac{\partial W}{\partial z} = 0,
\]

(1.56)

with \( \partial / \partial r = \nabla \). The transport equation (1.56) has the solution

\[
W(r, q; z) = W\left(r - \frac{2\pi q_t}{k} z, q; 0\right),
\]

(1.57)

which is equivalent to the result Eq. (1.44) in Section 1.6.1, with the special choice \( A = D = I \).

In a weakly inhomogeneous medium, the optical signal must satisfy the Helmholtz equation,

\[
-\frac{i}{k} \frac{\partial f}{\partial z} = \sqrt{k^2(r, z) + \frac{\partial^2}{\partial r^2}} f
\]

(1.58)

with \( k = k(r, z) \). In this case, we can again derive a transport equation for the Wigner distribution; the exact transport equation is rather complicated, but in the geometric-optical approximation, i.e., restricting ourselves to first-order derivatives, it takes the simple form

\[
\frac{2\pi q_t}{k} \frac{\partial W}{\partial r} + \sqrt{k^2 - \left(2\pi \right)^2 q_t^2 q} \frac{\partial W}{\partial z} + \left(\frac{\partial k}{2\pi \partial r}\right)^t \frac{\partial W}{\partial q} = 0,
\]

(1.59)

which, in general, cannot be solved explicitly. With the method of characteristics, however, we conclude that along a path defined by

\[
\frac{dr}{ds} = \frac{2\pi q_t}{k}, \quad \frac{dz}{ds} = \sqrt{k^2 - \left(2\pi \right)^2 q_t^2 q}, \quad \frac{dq}{ds} = \frac{\partial k}{2\pi \partial r},
\]

(1.60)

the Wigner distribution has a constant value. When we eliminate the frequency variable \( q \) from Eqs. (1.60), we are immediately led to

\[
\frac{d}{ds} \left(\frac{k}{ds} \frac{dr}{ds}\right) = \frac{\partial k}{\partial r}, \quad \frac{d}{ds} \left(\frac{k}{ds} \frac{dz}{ds}\right) = \frac{\partial k}{\partial z},
\]

(1.61)

which are the equations for an optical ray in geometrical optics.\(^71\) We are thus led to the general conclusion that in the geometric-optical approximation the Wigner distribution has a constant value along the geometric-optical ray paths, which is conform our conclusions in Section 1.6.4: \( W_o(r, q) \simeq W_i[g_x(r, q), g_y(r, q)] \). For a more detailed treatment of rays, we refer to Chapter 8 by Miguel Alonso.
1.7 Wigner distribution moments in first-order optical systems

The Wigner distribution moments provide valuable tools for the characterization of optical beams (see, for instance, Ref. 37). First-order moments, defined as

\[
[m_x, m_y, m_u, m_v] = \frac{1}{E} \iiint [x, y, u, v] W(x, y, u, v) \, dx \, dy \, du \, dv,
\]

yield the position of the beam \((m_x \text{ and } m_y)\) and its direction \((m_u \text{ and } m_v)\). Second-order moments, defined by Eq. (1.25), give information about the spatial width of the beam (the shape \(m_{xx} \text{ and } m_{yy}\) of the spatial ellipse and its orientation \(m_{xy}\)) and the angular width in which the beam is radiating (the shape \(m_{uu} \text{ and } m_{vv}\) of the spatial-frequency ellipse and its orientation \(m_{uv}\)). Moreover, they provide information about its curvature \((m_{xu} \text{ and } m_{yv})\) and its twist \((m_{xv} \text{ and } m_{yu})\), with a possible definition of the twistedness as

\[
\left( m_{xx}m_{uu} - m_{xu}^2 \right) + \left( m_{yy}m_{vv} - m_{yu}^2 \right) + 2 \left( m_{xy}m_{uv} - m_{xv}m_{yu} \right)
\]

(see also Section 1.7.1), are based on second-order moments. Also the longitudinal component of the orbital angular momentum \(\Lambda = \Lambda_a + \Lambda_v \propto (m_{xx} - m_{yy})\) [see Eq. (3) in Ref. 73] and its antisymmetrical part \(\Lambda_a\) and vortex part \(\Lambda_v\),

\[
\Lambda_a \propto \frac{(m_{xx} - m_{yy})(m_{xv} + m_{yu}) - 2m_{xy}(m_{xu} - m_{yv})}{m_{xx} + m_{yy}},
\]

\[
\Lambda_v \propto 2 \frac{m_{yy}m_{xv} - m_{xx}m_{yu} + m_{xy}(m_{xu} - m_{yv})}{m_{xx} + m_{yy}},
\]

[see Eqs. (22) and (21) in Ref. 73] are based on these moments.\(^{74}\) Higher-order moments are used, for instance, to characterize the beam’s symmetry and its sharpness.\(^ {37}\)

Because the Wigner distribution of a two-dimensional signal is a function of four variables, it is difficult to analyze. Therefore, the signal is often represented not by the Wigner distribution itself, but by its moments. Beam characterization based on the second-order moments of the Wigner distribution thus became the basis of an International Organization for Standardization standard.\(^ {75}\)

In this section we restrict ourselves mainly to second-order moments. The propagation of the matrix \(M\) of second-order moments of the Wigner distribution through a first-order optical system with ray transformation matrix \(T\), can be described by the input-output relationship\(^ {9,76}\) \(M_o = TM_i T^t\). This relationship can readily be derived by combining the input-output relationship (1.39) of the first-order optical system with the definition (1.25) of the moment matrices of the input and the output signal. Since the ray transformation matrix \(T\) is symplectic, we immediately conclude that a possible symplectivity of the moment matrix (to be discussed later) is preserved in a first-order optical system: if \(M_i\) is proportional to a symplectic matrix, then \(M_o\) is proportional to a symplectic matrix as well, with the same proportionality factor.
1.7.1 Moment invariants

If we multiply the moment relation $M_o = TM_i T^t$ from the right by $J$, and use the symplecticity property (1.41) and the properties of $J$, the input-output relationship can be written as $M_o J = T (M_i J) T^{-1}$. From the latter relationship we conclude that the matrices $M_i J$ and $M_o J$ are related to each other by a similarity transformation. As a consequence of this similarity transformation, and writing the matrix $MJ$ in terms of its eigenvalues and eigenvectors according to $MJ = S \Lambda S^{-1}$, we can formulate the relationships $\Lambda_o = \Lambda_i$ and $S_o = T S_i$. We are thus led to the important property that the eigenvalues of the matrix $MJ$ (and any combination of these eigenvalues) remain invariant under propagation through a first-order optical system, while the matrix of eigenvectors $S$ transforms in the same way as the ray vector $[x, q]^t$ does.

It can be shown that the eigenvalues of $MJ$ are real. Moreover, if $\lambda$ is an eigenvalue of $MJ$, then $-\lambda$ is an eigenvalue, too; this implies that the characteristic polynomial $\det(MJ - \lambda I)$, with the help of which we determine the eigenvalues, is a polynomial of $\lambda^2$. Indeed, the characteristic equation takes the form

$$\det(MJ - \lambda I) = 0 = \lambda^4 - a_2 \lambda^2 + a_4,$$

with $a_4 = \det M$ and

$$a_2 = (m_{xx} m_{uu} - m_{xu}^2) + (m_{yy} m_{uv} - m_{yu}^2) + 2(m_{xy} m_{uv} - m_{xu} m_{yu}).$$

Since the eigenvalues of $MJ$ are invariants, the same holds for the coefficients of the characteristic equation. And since the characteristic equation is an equation in $\lambda^2$, we have only two such independent eigenvalues ($\pm \lambda_x$ and $\pm \lambda_y$, say) and thus only two independent invariants (like, for instance, $\lambda_x$ and $\lambda_y$, or $a_2$ and $a_4$).

An interesting property follows from Williamson’s theorem: For any real, positive-definite symmetric matrix $M$, there exists a real symplectic matrix $T$ such that $M = T \Delta_o T^t$, where $\Delta_o = T^{-1} M (T^{-1})^t$ takes the normal form

$$\Delta_o = \begin{bmatrix} \Lambda_o & 0 \\ 0 & \Lambda_o \end{bmatrix}$$

with $\Lambda_o = \begin{bmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{bmatrix}$ and $\lambda_x, \lambda_y > 0$. (1.63)

From the similarity transformation $MJ = T_o (\Delta_o J) T_o^{-1}$, we conclude that $\Delta_o$ follows directly from the eigenvalues $\pm \lambda_x$ and $\pm \lambda_y$ of $MJ$ and that $T_o$ follows from the eigenvectors of $(MJ)^2$: $(MJ)^2 T_o = T_o \Delta_o^2$. Any moment matrix $M$ can thus be brought into the diagonal form $\Delta_o$ by means of a realizable first-order optical system with ray transformation matrix $T_o^{-1}$.

1.7.2 Moment invariants for phase-space rotators

In the special case that we are dealing with a phase-space rotator, for which the ray transformation matrix satisfies the orthogonality relation $T^{-1} = T^t$, we not only
have the similarity transformation $M_o \mathbf{J} = T (M_i \mathbf{J}) T^{-1}$ but also the similarity transformation $M_o = T M_i T^{-1}$. The eigenvalues of $M$ are now also invariants, and the same holds for the coefficients of the corresponding characteristic equation

$$\det (M - \mu \mathbf{I}) = 0 = \mu^4 - b_1 \mu^3 + b_2 \mu^2 - b_3 \mu + b_4.$$  

Since $b_4 = \det M$ is already a known invariant ($= a_4$), this yields at most three new independent invariants.

Another way to find moment invariants for phase-space rotators is to consider the Hermitian matrix

$$M' = \frac{1}{E} \int \int (r - i q) (r - i q)^\dagger W(r, q) \, dr \, dq = M_{rr} + M_{qq} + i(M_{rq} - M_{qr}^t) = \begin{bmatrix} m_{xx} + m_{uu} & m_{xy} + m_{uv} + i(m_{xv} - m_{yu}) \\ m_{xy} + m_{uv} - i(m_{xv} - m_{yu}) & m_{yy} + m_{vv} \end{bmatrix} = \begin{bmatrix} Q_0 + Q_1 & Q_2 + iQ_3 \\ Q_2 - iQ_3 & Q_0 - Q_1 \end{bmatrix},$$

and to use Eq. (1.50) to get the relation

$$M'_o = U M'_i U^\dagger = U M'_i U^{-1},$$

which is again a similarity transformation. Note that the moments $m_{xy}$ and $m_{uv}$, i.e., the diagonal entries of the submatrix $M_{rq}$, do not enter the matrix $M'$ and that we have introduced the four moment combinations $Q_j (j = 0, 1, 2, 3)$ as

$$Q_0 = \frac{1}{2} [(m_{xx} + m_{uu}) + (m_{yy} + m_{vv})],$$

$$Q_1 = \frac{1}{2} [(m_{xx} + m_{uu}) - (m_{yy} + m_{vv})],$$

$$Q_2 = m_{xy} + m_{uv},$$

$$Q_3 = m_{xv} - m_{yu}. $$

The characteristic equation with which the eigenvalues of $M'$ can be determined, reads

$$\det (M' - \nu \mathbf{I}) = 0 = \nu^2 - 2Q_0 \nu + Q_0^2 - Q^2 = (\nu - Q_0)^2 - Q^2,$$

where we have also introduced

$$Q = \sqrt{Q_1^2 + Q_2^2 + Q_3^2}.$$  

The eigenvalues are real and we can write $\nu_1, 2 = Q_0 \pm Q$. Since the eigenvalues are invariant, we immediately get that $\nu_1 - \nu_2 = 2Q$ is an invariant, and we also get the invariants $\nu_1 + \nu_2 = 2Q_0 = b_1$, which is the trace of $M'$ and of $M$, and $\nu_1 \nu_2 =$
$Q_0^2 - Q_2^2 = b_2 - a_2$, which is the determinant of $M'$. We remark that $Q_3$ corresponds to the longitudinal component of the orbital angular momentum of a paraxial beam propagating in the $z$ direction. From the invariance of $Q$, we conclude that the three-dimensional vector $(Q_1, Q_2, Q_3) = (Q \cos \vartheta, Q \sin \vartheta \cos \gamma, Q \sin \vartheta \sin \gamma)$ lives on a sphere with radius $Q$. It is not difficult to show now that $M'$ can be represented in the general form

$$M' = Q_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + Q \begin{bmatrix} \cos \vartheta & \exp(i \gamma) \sin \vartheta \\ \exp(-i \gamma) \sin \vartheta & -\cos \vartheta \end{bmatrix},$$

(1.67)

where the angles $\vartheta$ and $\gamma$ follow from the relations $Q \cos \vartheta = Q_1$ (with $0 \leq \vartheta \leq \pi$) and $Q \exp(i \gamma) \sin \vartheta = Q_2 + iQ_3$.

A phase-space rotator will only change the values of the angles $\vartheta$ and $\gamma$, but does not change the invariants $Q_0$ and $Q$. To transform a diagonal matrix $M'$ with diagonal entries $Q_0 + Q$ and $Q_0 - Q$ into the general form (1.67), we can use, for instance, the phase-space rotating system\(^{31}\) $F(\frac{1}{2} \gamma_x, -\frac{1}{2} \gamma_y) R(-\frac{1}{2} \vartheta) F(-\frac{1}{2} \gamma_x, \frac{1}{2} \gamma_y)$, see also Section 1.6.2 and Eq. (1.48). Moreover, from Eq. (1.65), we easily derive\(^{30}\) that for a separable fractional Fourier transformer $F(\gamma_x, \gamma_y)$, $Q_1$ is an invariant and $Q_2 + iQ_3$ undergoes a rotation-type transformation: $(Q_2 + iQ_3)_o = \exp[i(\gamma_x - \gamma_y)] (Q_2 + iQ_3)_i$. Similar properties hold for a gyrator $G(\varphi)$, for which $Q_2$ is an invariant and $(Q_3 + iQ_1)_o = \exp(i2\varphi) (Q_3 + iQ_1)_i$, and for a rotator $R(-\varphi)$, for which $Q_3$ is an invariant and $(Q_1 + iQ_2)_o = \exp(i2\varphi) (Q_1 + iQ_2)_i$.

### 1.7.3 Symplectic moment matrix – the bilinear ABCD law

If the moment matrix $M$ is proportional to a symplectic matrix, it can be expressed in the form\(^{77}\)

$$M = m \begin{bmatrix} G^{-1} & G^{-1}H \\ HG^{-1} & G + HG^{-1}H \end{bmatrix},$$

(1.68)

with $m$ a positive scalar, $G$ and $H$ real symmetric $2 \times 2$ matrices, and $G$ positive-definite; the two positive eigenvalues of $MJ$ are now equal to $+m$ and the two negative eigenvalues are equal to $-m$.

We recall that for a symplectic moment matrix, the input-output relation $M_o = TM_o T'$ can be expressed equivalently in the form of the bilinear relationship

$$H_o \pm iG_o = [C + D(H_i \pm iG_i)][A + B(H_i \pm iG_i)]^{-1}.$$

(1.69)

This bilinear relationship, together with the invariance of $\det M$, completely describes the propagation of a symplectic matrix $M$ through a first-order optical system. Note that the bilinear relationship (1.69) is identical to the ABCD-law for spherical waves: for spherical waves we have $H_o = |C + DH_i|A + BH_i|^{-1}$, and we have only replaced the (real) curvature matrix $H$ by the (generally complex) matrix $H \pm iG$. We are thus led to the important result that if the matrix $M$ of second-order moments is symplectic (up to a positive constant) as described in Eq. (1.68), its propagation through a first-order optical system is completely described by the invariance of this positive constant and the ABCD-law (1.69).
1.7.4 Measurement of the moments

Several optical schemes to determine all ten second-order moments have been described.\textsuperscript{72,82–87} We mention in particular Ref. 87, which is based on a general scheme that can also be used for the determination of arbitrary higher-order moments, $\mu_{pqrs}$, with

$$
\mu_{pqrs} E = \iiint W(x, y, u, v) x^p u^q y^r v^s \, dx \, dy \, du \, dv \quad (p, q, r, s \geq 0);
$$

(1.70)

note that for $q = s = 0$ we have intensity moments,

$$
\mu_{p0r0} E = \iiint W(x, y, u, v) x^p y^r \, dx \, dy \, du \, dv = \iint x^p y^r \Gamma(x, x; y, y) \, dx \, dy \quad (p, r \geq 0),
$$

(1.71)

which can easily be measured. The ten second-order moments can be determined from the knowledge of the output intensities of four first-order optical systems, where one of them has to be anamorphic. For the determination of the 20 third-order moments, for instance, we thus find the need of using a total of six first-order optical systems: four isotropic systems and two anamorphic systems. For the details of how to construct appropriate measuring schemes, we refer to Ref. 87.

1.8 Coherent signals and the Cohen class

The Wigner distribution belongs to a broad class of space-frequency functions known as the Cohen class.\textsuperscript{30} Any function of this class is described by the general formula

$$
C_f(r, q) = \iiint f(r_0 + \frac{1}{2} r') f^*(r_0 - \frac{1}{2} r') k(r, q, r', q')
\times \exp[-i2\pi(q' r' - r' q' + r_0^q q_0^q)] \, dr' \, dq' \, dr_0 \, dq_0
$$

(1.72)

and the choice of the kernel $k(r, q, r', q')$ selects one particular function of the Cohen class. The Wigner distribution, for instance, arises for $k(r, q, r', q') = 1$, whereas $k(r, q, r', q') = \delta(r - r') \delta(q - q')$ yields the ambiguity function. In this chapter we will restrict ourselves to the case that $k(r, q, r', q')$ does not depend on the space variable $r$ and the spatial-frequency variable $q$, hence $k(r, q, r', q') = K(r', q')$, in which case the resulting space-frequency distribution is shift covariant, see Section 1.4.2.

1.8.1 Multi-component signals – auto-terms and cross-terms

The Wigner distribution, like the mutual coherence function and the cross-spectral density, is a bilinear signal representation. In the case of completely coherent light,
However, we usually deal with a linear signal representation. Using a bilinear representation to describe coherent light thus yields cross-terms if the signal consists of multiple components. The two-component signal \( f(r) = f_1(r) + f_2(r) \) yields the Wigner distribution

\[
W_f(r, q) = W_{f_1}(r, q) + W_{f_2}(r, q) + 2 \Re \left\{ \int f_1(r + \frac{1}{2}r') f_2^*(r - \frac{1}{2}r') \exp(-i2\pi q'r') \, dr' \right\}
\]

(1.73)

and we notice a cross-term in addition to the auto-terms \( W_{f_1}(r, q) \) and \( W_{f_2}(r, q) \). In the case of two point sources \( \delta(r - r_1) \) and \( \delta(r - r_2) \), for instance, the cross-term reads

\[
2 \delta[r - \frac{1}{2}(r_1 + r_2)] \cos[2\pi(r_1 - r_2)^t q)].
\]

It appears at the position \( \frac{1}{2}(r_1 + r_2) \), i.e., in the middle between the two auto-terms \( W_{f_1}(r, q) = \delta(r - r_1) \) and \( W_{f_2}(r, q) = \delta(r - r_2) \), and is modulated in the \( q \) direction. We can get rid of this cross-term when we average the Wigner distribution with a kernel that is narrow in the \( r \) direction and broad in the \( q \) direction. We thus remove the cross-term without seriously disturbing the auto-terms.

The occurrence of cross terms is also visible from the general condition

\[
W_f(r + \frac{1}{2}r'', q + \frac{1}{2}q'') W_f(r - \frac{1}{2}r'', q - \frac{1}{2}q'')
= \int \int W_f(r + \frac{1}{2}r'', q + \frac{1}{2}q'') W_f(r - \frac{1}{2}r'', q - \frac{1}{2}q') \times \exp[-i2\pi(q''r' - q''r')]) \, dr' \, dq',
\]

(1.74)

which, for \( r'' = q'' = 0 \), reduces to

\[
W_f^2(r, q) = \int \int W_f(r + \frac{1}{2}r', q + \frac{1}{2}q') W_f(r - \frac{1}{2}r', q - \frac{1}{2}q') \, dr' \, dq'.
\]

(1.75)

From the latter equality we conclude that the value of the Wigner distribution at some phase-space point \( (r, q) \) is related to the values of all those pairs of points \( (r \pm \frac{1}{2}r', q \pm \frac{1}{2}q') \) for which \( (r, q) \) is the midpoint. Using, as we generally do, the analytic signal \( f(r) \) instead of the real signal \( \frac{1}{2}[f(r) + f^*(r)] \), avoids the cross-terms that otherwise would automatically appear around \( q = 0 \).

The requirement of removing cross-terms without seriously affecting the auto-terms has led to the Cohen class of bilinear signal representations. All members \( C_f(r, q) \) of this class can be generated by a convolution (both for \( r \) and \( q \)) of the Wigner distribution with an appropriate kernel \( K(r, q) \):

\[
C_f(r, q) = K(r, q) * * W_f(r, q) = \int \int K(r - r_o, q - q_o) W_f(r_o, q_o) \, dr_o \, dq_o.
\]

(1.76)
Note that a convolution keeps the important property of shift covariance! After Fourier transforming the latter equation, we are led to an equation in the ‘ambiguity domain,’ and the convolution becomes a product:

$$\bar{C}_f(r', q') = \bar{K}(r', q') A_f(r', q')$$ \hspace{1cm} (1.77)

with

$$\bar{C}_f(r', q') = \mathcal{F}[C_f(r, q)](r', q')$$ \hspace{1cm} (1.78a)
$$A_f(r', q') = \mathcal{F}[W_f(r, q)](r', q')$$ \hspace{1cm} (1.78b)
$$\bar{K}(r', q') = \mathcal{F}[K(r, q)](r', q').$$ \hspace{1cm} (1.78c)

The product form (1.77) offers an easy way in the design of appropriate kernels.

Again, cf. Fig. 1.1, we position the different signal and kernel representations at the corners of a rectangle, see Fig. 1.4. For completeness we have also introduced the kernels \(R(r_1, r_2)\) and \(\bar{R}(q_1, q_2)\) that operate on the product \(\Gamma_f(r_1, r_2) = f(r_1) f_2(r_2)\) and \(\Gamma_f(q_1, q_2) = f(q_1) f_2(q_2)\), respectively, by means of a convolution for \(r\) or \(q\). Again, we have Fourier transformations along the sides of the rectangle, and we readily see that the kernel \(K(r, q)\) is related to the kernels \(R(r_1, r_2)\) and \(\bar{R}(q_1, q_2)\) as

$$K(r, q) = \int R(r + \frac{1}{2} r', r - \frac{1}{2} r') \exp(-i 2\pi q r') \, dr',$$ \hspace{1cm} (1.79a)
$$\bar{K}(r, q) = \int \bar{R}(q + \frac{1}{2} q', q - \frac{1}{2} q') \exp(i 2\pi r q') \, dq'.$$ \hspace{1cm} (1.79b)

As an example, we mention that the kernel \(K(r, q) = \delta(r) \delta(q)\), for which

$$C_f(r, q) = K(r, q) * \mathcal{F}^{-1}[W_f(r, q)]$$
$$C_f(r', q') = \bar{K}(r', q') A_f(r', q')$$
$$\bar{R}(q + \frac{1}{2} q, q - \frac{1}{2} q) * \mathcal{F}^{-1}[\bar{W}_f(q + \frac{1}{2} q, q - \frac{1}{2} q)]$$

Figure 1.4 Schematic representation of the cross-spectral density \(\Gamma\), its spatial Fourier transform \(\bar{\Gamma}\), the Wigner distribution \(W\), and the ambiguity function \(A\), together with the corresponding kernels \(R\), \(\bar{R}\), \(K\), and \(\bar{K}\), on a rectangle.

\(C_f(r, q) = W_f(r, q)\) is the Wigner distribution, corresponds to the kernels \(\bar{K}(r', q') = 1, R(r + \frac{1}{2} r, r' - \frac{1}{2} r) = \delta(r),\) and \(\bar{R}(q + \frac{1}{2} q, q - \frac{1}{2} q) = \delta(q)\).
1.8.2 One-dimensional case and some basic Cohen kernels

Many kernels have been proposed in the past, and some already existing bilinear signal representations have been identified as belonging to the Cohen class with an appropriately chosen kernel. Table 1.2 mentions some of them.\(^{30,31,36}\)

<table>
<thead>
<tr>
<th>bilinear signal representation</th>
<th>(\tilde{K}(x',u'))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wigner (W(x,u)), Eq. (1.14)</td>
<td>1</td>
</tr>
<tr>
<td>pseudo-Wigner (P(x,u;w)), Eq. (1.17)</td>
<td>(w(\frac{1}{2}x')) (w^*(-\frac{1}{2}x'))</td>
</tr>
<tr>
<td>Page</td>
<td>(\exp(-i\pi u'</td>
</tr>
<tr>
<td>Kirkwood-Rihaczek</td>
<td>(\exp(-i\pi u'x'))</td>
</tr>
<tr>
<td>(w)-Rihaczek</td>
<td>(w(x')) (\exp(-i\pi u'x'))</td>
</tr>
<tr>
<td>Levin</td>
<td>(\cos(\pi u'x'))</td>
</tr>
<tr>
<td>(w)-Levin</td>
<td>(w(x')) (\cos(\pi u'x'))</td>
</tr>
<tr>
<td>Born-Jordan (sinc)</td>
<td>(\sin(\alpha \pi u'x')/\alpha \pi u'x')</td>
</tr>
<tr>
<td>Zhao-Atlas-Marks (cone/windowed sinc)</td>
<td>(w(x')</td>
</tr>
<tr>
<td>Choi-Williams (exponential)</td>
<td>(\exp[-(u'x')^2/\sigma])</td>
</tr>
<tr>
<td>spectrogram (</td>
<td>S(x,u;w)</td>
</tr>
<tr>
<td></td>
<td>(A_w(-x',-u'))</td>
</tr>
</tbody>
</table>

**Table 1.2** Kernels \(\tilde{K}(x', u')\) of some basic Cohen-class bilinear signal representations.

In designing kernels, one may try to keep the interesting properties of the Wigner distribution; this reflects itself in conditions for the kernel. We recall that shift covariance is already maintained. To keep also the properties of realness, \(x\) marginal, and \(u\) marginal, for instance, the kernel \(\tilde{K}(x', u')\) should satisfy the conditions \(\tilde{K}(x', u') = \tilde{K}^*(-x', -u')\), \(\tilde{K}(0, u') = 1\), and \(\tilde{K}(x', 0) = 1\), respectively. To keep the important property that for a signal \(f(x) = |f(x)|\exp[i2\pi\phi(x)]\) the instantaneous frequency \(d\phi/dx\) should follow from the bilinear representation through

\[
\frac{d\phi}{dx} = \frac{\int u C_f(x, u) \, du}{\int C_f(x, u) \, du},
\]

like it does for the Wigner distribution, the kernel should satisfy the condition

\[
\tilde{K}(0, u') = \text{constant} \quad \text{and} \quad \frac{\partial \tilde{K}}{\partial x'} \bigg|_{x' = 0} = 0.
\]
The Levin, Born-Jordan, and Choi-Williams representations clearly satisfy these conditions.

1.8.3 Rotation of the kernel

In the case of two point sources $\delta(x-x_1)$ and $\delta(x-x_2)$, the cross-term

$$2 \delta[x - \frac{1}{2}(x_1 + x_2)] \cos[2\pi(x_1 - x_2)u]$$

was located such that we needed averaging in the $u$ direction when we want to remove it. In other cases, the cross-term may be located such that we need averaging in a different direction; for two plane waves $\exp(i2\pi u_1 x)$ and $\exp(i2\pi u_2 x)$, for instance, the cross-term reads

$$2 \delta[u - \frac{1}{2}(u_1 + u_2)] \cos[2\pi(u_1 - u_2)x]$$

and we need averaging in the $x$ direction. We may thus benefit from a rotation of the kernel, or let the original kernel operate on the Wigner distribution of the fractional Fourier transform of the signal,

$$C_f(x, u) = K(x \cos \gamma + u \sin \gamma, -x \sin \gamma + u \cos \gamma) * W_f(x, u), \quad (1.80a)$$

$$C_{F_\gamma}(x, u) = K(x, u) * W_{F_\gamma}(x, u). \quad (1.80b)$$

To find the optimal rotation angle $\gamma_0$, one may proceed as follows. Let $m_x^\gamma$ and $m_{xx}^\gamma$ be the first- and second-order moments of the intensity $|F_\gamma(x)|^2$ of the fractional Fourier transform $F_\gamma(x)$,

$$m_x^\gamma = \frac{1}{E} \int \int x W_{F_\gamma}(x, u) \, dx \, du = \frac{1}{E} \int x |F_\gamma(x)|^2 \, dx, \quad (1.81a)$$

$$m_{xx}^\gamma = \frac{1}{E} \int \int x^2 W_{F_\gamma}(x, u) \, dx \, du = \frac{1}{E} \int x^2 |F_\gamma(x)|^2 \, dx, \quad (1.81b)$$

and let $m_{xu}^\gamma$ be the mixed moment

$$m_{xu}^\gamma = \frac{1}{E} \int \int xu W_{F_\gamma}(x, u) \, dx \, du. \quad (1.81c)$$

The propagation laws for the first- and second-order moments through a rotator read

$$\begin{bmatrix} m_x^\gamma \\ m_u^\gamma \end{bmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{bmatrix} \begin{bmatrix} m_x \\ m_u \end{bmatrix}, \quad (1.82a)$$

$$\begin{bmatrix} m_{xx}^\gamma & m_{xu}^\gamma \\ m_{xu}^\gamma & m_{uu}^\gamma \end{bmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{bmatrix} \begin{bmatrix} m_{xx} \\ m_{xu} \\ m_{ux} \\ m_{uu} \end{bmatrix} \begin{bmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{bmatrix}. \quad (1.82b)$$

Note that $m_u = m_x^{\pi/2}$, $m_{uu} = m_{xx}^{\pi/2}$, and $m_{xu} = m_{xx}^{\pi/4} - \frac{1}{2}(m_{xx} + m_{xx}^{\pi/2})$, and that all second-order moments follow directly from the measurement of the intensity
1.8. Coherent signals and the Cohen class

profiles of only three fractional Fourier transforms: $F_0(x) = f(x)$, $F_{\pi/2}(x) = \bar{f}(x)$, and $F_{\pi/4}(x)$. While the second-order moment $m_{\gamma xx}^2$ can be expressed as

$$ m_{\gamma xx} = m_{xx} \cos^2 \gamma + m_{uu} \sin^2 \gamma + m_{xu} \sin 2\gamma, \quad (1.83) $$

the second-order central moment $\mu_{\gamma xx} = m_{\gamma xx} - (m_{\gamma x}^2)$ can be expressed as

$$ \mu_{\gamma xx} = \mu_{xx} \cos^2 \gamma + \mu_{uu} \sin^2 \gamma + (m_{xu} - m_{xx} m_{uu}) \sin 2\gamma, \quad (1.84) $$

and extremum values of $\mu_{\gamma xx}$ arise for the angle $\gamma_0$, defined by

$$ \tan 2\gamma_0 = \frac{2 m_{xu} - m_{xx} m_{uu}}{\mu_{xx} - \mu_{uu}} = \frac{\frac{\pi}{4}}{2} \left( \frac{m_{0 xx}^2 + m_{0 xx}^2}{m_{xx}^2 - m_{uu}^2 - (m_{xx})^2} + \left( \frac{m_{xx}^2}{2} \right)^2 \right). \quad (1.85) $$

Note that $\gamma_0$ corresponds to the minimum value of $\mu_{\gamma xx}$, if $\gamma_0$ is chosen such that $\cos 2\gamma_0$ has the same sign as $\mu_{\pi xx} - \mu_{0 xx}$; $\gamma_0 + \frac{\pi}{2}$ then corresponds to the maximum value of $\mu_{\gamma xx}$. The angles $\gamma_0$ and $\gamma_0 + \frac{\pi}{2}$ determine the principal axes of the moment ellipse in phase space. Kernels can be optimized by rotating them and aligning them to these principal axes.\textsuperscript{89}

1.8.4 Rotated version of the smoothed interferogram

We will apply the aligning of the kernel to the smoothed interferogram, which can best be derived from the pseudo-Wigner distribution. With

$$ S_f(x, u; w) = \int f(x + x_0) w^*(x_0) \exp(-i2\pi ux_0) dx_0 \quad (1.86) $$

denoting the windowed Fourier transform, the pseudo-Wigner distribution $P_f(x, u; w)$, i.e., the Wigner distribution with the additional window $w\left(\frac{1}{2} x'\right) w^*\left(-\frac{1}{2} x'\right)$ in its defining integral, see Eq. (1.17), can also be represented as

$$ P_f(x, u; w) = \int S_f(x, u + \frac{1}{2} t; w) S_f^*(x, u - \frac{1}{2} t; w) dt. \quad (1.87) $$

The smoothed interferogram, also known as the S-method, is now defined as\textsuperscript{90}

$$ P_f(x, u; w, z) = \int S_f(x, u + \frac{1}{2} t; w) z(t) S_f^*(x, u - \frac{1}{2} t; w) dt; \quad (1.88) $$

it is based on the pseudo Wigner distribution written in the form (1.87), but with an additional smoothing window $z(t)$ in the $u$ direction. The resulting distribution is of the Wigner distribution form, with significantly reduced cross-terms of multi-component signals, while the auto-terms are close to those in the pseudo-Wigner distribution. For $z(t) = \delta(t)$, the bilinear representation $P_f(x, u; w, z) = |S_f(x, u; w)|^2$ is known as the spectrogram: the squared modulus.
Chapter 1. Wigner Distribution in Optics

of the windowed Fourier transform. For $z(t) = 1$, $P_f(x, u; w, z)$ reduces to the pseudo-Wigner distribution (1.87).

Since the window $z(t)$ controls the behavior of $P_f(x, u; w, z)$ – more Wigner-type or more spectrogram-type – we spend one paragraph on the spectrogram. Although the spectrogram is a quadratic signal representation, $|S_f(x, u; w)|^2$, the squaring is introduced only in the final step and therefore does not lead to undesirable cross-terms that are present in other bilinear signal representations. This freedom from artifacts, together with simplicity, robustness, and ease of interpretation, has made the spectrogram a popular tool for speech analysis since its invention in 1946.91 The price that has to be paid, however, is that the auto-terms are smeared by the window $w(x)$. Note that for $w(x) = \delta(x)$, the spectrogram yields the pure space representation $|S_f(x, u; w)|^2 = |f(x)|^2$, whereas for $w(x) = 1$, it yields the pure frequency representation $|S_f(x, u; w)|^2 = |\bar{f}(u)|^2$. This has been illustrated in Fig. 1.5 on the sinusoidal FM signal

$$\exp\{i[2\pi u_0 x + a_1 \sin(2\pi u_1 x)]\}$$

and a rectangular window $w(x) = \text{rect}(x/X)$ of variable width $X$. Note in particular the smearing that appears in Fig. 1.5a.

**Figure 1.5** Spectrogram of a sinusoidal FM signal $\exp\{i[2\pi u_0 x + a_1 \sin(2\pi u_1 x)]\}$ with (a) a medium-sized window, leading to a space-frequency representation with smearing, and (b) a long window, leading to a pure frequency representation.

Based on Eq. (1.88), but replacing the signal $f(x)$ by its fractional Fourier transform $F_\gamma(x)$, the $\gamma$-rotated version $P^\gamma_f(x, u; w, z)$ of the smoothed interferogram $P_f(x, u; w, z)$ was defined subsequently as89,92

$$P^\gamma_f(x, u; w, z) = P_{F_\gamma}(x, u; w, z)$$

$$= \int S_{F_\gamma}(x, u + \frac{1}{2}t; w) z(t) S_{F_\gamma}^*(x, u - \frac{1}{2}t; w) dt. \quad (1.89)$$
A definition directly in terms of the signal $f(x)$ reads

$$P_f^\gamma(x, u; w, z) = \int S_f(x + \frac{1}{2}t \sin \gamma, u + \frac{1}{2}t \cos \gamma; W_\gamma) z(t) \exp(-i2\pi ut \sin \gamma)$$

$$\times S_f^*(x - \frac{1}{2}t \sin \gamma, u - \frac{1}{2}t \cos \gamma; W_\gamma) dt,$$  

(1.90)

where the fractional Fourier transform $W_\gamma(x)$ of the window $w(x)$ arises and

where we have used the relationship

$$S_{f\gamma}(x_2, u_2; W_\gamma) = \exp[i\pi(u_2x_2 - u_1x_1)] S_f(x_1, u_1; w)$$

(1.91)

with

$$\begin{bmatrix} x_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{bmatrix} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix}.$$  

The $\gamma$-rotated smoothed interferogram $P_f^\gamma(x, u; w, z)$ is related to the Wigner distribution $W_f(x, u)$ with the kernels$^{89,92}$

$$K(x, u) = W_w(-x, -u) \bar{z}(-x \cos \gamma + u \sin \gamma),$$  

(1.92a)

$$K(x', u') = \int A_w(-x' + t \sin \gamma, -u' + t \cos \gamma) z(t) dt.$$  

(1.92b)

Note that for $\gamma = 0$, the distribution $P_f^\gamma(x, u; w, z)$ reduces to the one originally introduced, which was based on a combination of windowed Fourier transforms in the $u$ direction, while for $\gamma = \frac{1}{2}\pi$ it reduces to the version that combines these windowed Fourier transforms in the $x$ direction.$^{90}$

The rotated version of the smoothed interferogram is a versatile method to remove cross-terms. To illustrate this, we show two numerical examples. Consider first the signal

$$f(x) = \exp[-(3x/x_o)^6] \{ \exp[i\phi_1(x)] + \exp[i\phi_2(x)] \}$$

with

$$\begin{align*}
\phi_1(x) &= \pi h_1 x^2 + a_1 \cos(2\pi u_1 x), \\
\phi_2(x) &= \pi h_2 x^2 + a_2 \cos(2\pi u_2 x),
\end{align*}$$

consisting of two components with instantaneous frequency $h_1 x - a_1 u_1 \sin(2\pi u_1 x)$ and $h_2 x - a_2 u_2 \sin(2\pi u_2 x)$, respectively; note that the instantaneous frequencies cross at $x = 0$. In the numerical simulation, the variables take the values $x_o = 128$, $h_1 = 192$, $h_2 = 64$, $u_1 = u_2 = 2$, $-a_1 = a_2 = 8/\pi$. A Hann(ing) window $w(x) = \cos^2(\pi x/X) \text{rect}(x/X)$ with width $X = 128$, is used for the calculation of the windowed Fourier transform $S_f(x, u; w)$. The values of the normalized second-order central moments are $\mu_{xx}^0 = 1$, $\mu_{xx}^{3/2} = 1.38$, and $\mu_{xx}^{\pi/4} = 0.07$. According to Eq. (1.85), and using the fact that $\mu_{xx}^{3/2} - \mu_{xx}^0 > 0$, we get $\gamma_0 = 41^\circ$. The second-order moment in this direction, $\mu_{xx}^{41^\circ} = 0.057$, is smaller than in any other direction, while the second-order moment in the orthogonal direction,
The fractional Fourier transform $F_{\gamma}(x)$ of the signal $f(x)$ for the angle $\gamma = \gamma_0 - \frac{1}{2}\pi = -49^\circ$ can now be calculated by using a discrete fractional Fourier transformation algorithm. The next step is to calculate the windowed Fourier transform $S_{F_{\gamma}}(x, u; w)$ of the fractional Fourier transform $F_{\gamma}(x)$ and to use it in Eq. (1.89).

The results of this analysis are presented in Fig. 1.6. The pseudo-Wigner distribution $P_f(x, u; w)$ is shown in Fig. 1.6a. The smoothed interferogram $P_f(x, u; w, z)$, calculated by the standard definition (1.88), i.e., combining terms along the $u$ axis, with a rectangular window $z(t) = \text{rect}(t/T)$ and $T = 15$, is presented in Fig. 1.6b. We see that some cross-terms already appear, although the auto-terms are still very different from those in the Wigner distribution in Fig. 1.6a. The reason lies in the very significant spread of one component along the $u$ axis. The $\gamma$-rotated smoothed interferogram $P^\gamma_f(x, u; w, z)$ for the optimal fractional angle $\gamma = -49^\circ$ is presented in Fig. 1.6c for a rectangular window with $T = 9$ and in Fig. 1.6d for a Hann(ing) window with $T = 15$. We can see that, as a consequence of the high concentration of the components along the optimal fractional angle, we almost achieved the goal of getting the auto-terms of the Wigner distribution without any cross-terms.
1.9 Conclusion

We have presented an overview of the Wigner distribution and of some of its properties and applications in an optical context. The Wigner distribution describes a signal in space (i.e., position) and spatial frequency (i.e., direction), simultaneously, and can be considered as the local frequency spectrum of the signal, like the score in music and the phase space in mechanics. Although derived in terms of Fourier optics, the description of a signal by means of its Wigner distribution closely resembles the ray concept in geometrical optics. It thus presents a link between Fourier optics and geometrical optics. Moreover, the concept of the Wigner distribution is not restricted to deterministic signals (i.e., completely

Similar results are obtained for the signal

\[ f(x) = \exp[-(3x/x_0)^8] \{ \exp[i(\phi(x) + 50\pi x)] + \exp[i(\phi(x) - 50\pi x)] \} \]

with \( \phi(x) = \int_{-\infty}^{x} 15\pi \arcsinh(100 \xi) \, d\xi, \)

where \( x_0 = X = 128 \) again and \( T = 21 \), see Fig. 1.7.
coherent light); it can be applied to stochastic signals (i.e., partially coherent light), as well, thus presenting a link between partial coherence and radiometry.

Properties of the Wigner distribution and its propagation through linear systems have been considered; the corresponding description of signals and systems can directly be interpreted in geometric-optical terms. For first-order optical systems, the propagation of the Wigner distribution is completely determined by the system’s ray transformation matrix, thus presenting a strong interconnection with matrix optics.

We have studied the second-order moments of the Wigner distribution and some interesting combinations of these moments, together with the propagation of these moment combinations through first-order optical systems. Special attention has been paid to systems that perform rotations in phase space.

In the case of completely coherent light, the Wigner distribution is a member of a broad class of bilinear signal representations, known as the Cohen class. Each member of this class is related to the Wigner distribution by means of a convolution with a certain kernel. Because of the quadratic nature of such signal representations, they suffer from unwanted cross-terms, which one tries to minimize by a proper choice of this kernel. Some members of the Cohen class have been reviewed, and special attention was devoted to the smoothed interferogram in combination with the optimal angle in phase space in which the smoothing takes place.

References


55. T. Alieva and M. J. Bastiaans, “Phase-space distributions in quasi-polar
    2329 (2000).
    close fractional Fourier power spectra,” IEEE Trans. Signal Process. 51, 112–
57. M. J. Bastiaans and K. B. Wolf, “Phase reconstruction from intensity
    Berkeley, CA (1966).
60. S. A. Collins, Jr., “Lens-system diffraction integral written in terms of matrix
61. M. Moshinsky and C. Quesne, “Linear canonical transformations and their
63. O. Bryngdahl, “Geometrical transformations in optics,” J. Opt. Soc. Am. 64,
64. J.-Z. Jiao, B. Wang, and H. Liu, “Wigner distribution function and geometrical
67. H. Bremmer, “General remarks concerning theories dealing with scattering
68. J. J. McCoy and M. J. Beran, “Propagation of beamed signals through
    1149 (1976).
69. I. M. Besieris and F. D. Tappert, “Stochastic wave-kinetic theory in the


